Summary: Exercises regarding FK formula and the BS model

11. Solve the following differential equation:

$$\frac{\partial F(t,x)}{\partial t} + \underset{\downarrow \, \mu}{rx}\frac{\partial F(t,x)}{\partial x} + \underset{\downarrow \, \frac{1}{2}\sigma^2}{\frac{1}{2}\sigma^2 x^2}\frac{\partial^2 F(t,x)}{\partial x^2} = 0$$

for $t \in [0,T]$, such that $F(T,x) = \frac{\ln x^2 + K}{2}$.

"auxiliary" process:

$$dX(\tau) = R\,X(\tau)\,d\tau + \sigma X(\tau)\,dW(\tau)$$

$\Rightarrow$ i.e., $\{X(\tau),\ t \in [0,T]\}$ is a Geometric Brownian Motion

$$X(T) = X(\tau)\, e^{(R - \frac{1}{2}\sigma^2)(T-\tau) + \sigma(W(T) - W(\tau))}$$

According to the Feynman-Kac formula:

$$F(\tau, x) = E[\,F(T, X(T)) \mid X(\tau) = x\,]$$

$$= E\left[\frac{\ln(X(T)^2) + K}{2} \;\middle|\; X(\tau) = x\right] = \circledast$$

$$\ln(X^2(T)) = \ln\left(X^2(\tau)\, e^{2(R - \frac{1}{2}\sigma^2)(T-\tau) + 2\sigma(W(T) - W(\tau))}\right)$$

$$= 2\ln X(\tau) + 2(R - \tfrac{1}{2}\sigma^2)(T-\tau) + 2\sigma(W(T) - W(\tau))$$

$$\circledast = E\Big[\ln \underbrace{X(\tau)}_{x} + (R - \tfrac{1}{2}\sigma^2)(T-\tau) + \sigma(W(T) - W(\tau)) + \frac{K}{2} \;\Big|\; X(\tau) = x\Big]$$

$$= \ln x + (R - \tfrac{1}{2}\sigma^2)(T-\tau) + \sigma E[\underbrace{W(T) - W(\tau)}_{\perp\!\!\!\perp\, X(\tau)} \mid X(\tau) = x] + \frac{K}{2}$$

$$= \ln x + (r - \tfrac{1}{2}\sigma^2)(T-\tau) + \sigma \underbrace{E[W(T) - W(\tau)]}_{=0} + \frac{K}{2}$$

ie:

$$F(\tau, x) = \ln x + \left(r - \tfrac{1}{2}\sigma^2\right)(T-\tau) + \frac{k}{2}$$

$$\frac{\partial F(\tau, x)}{\partial \tau} + \underbrace{r}_{\mu} \frac{\partial F}{\partial x} + \underbrace{\tfrac{1}{2}\sigma^2}_{\frac{1}{2}\sigma^2} \frac{\partial^2 F}{\partial x^2} = 0$$

$$dx(\tau) = R\,dt + \sigma\,dw(\tau)$$

$$\Rightarrow X(T) = x(\tau) + \int_\tau^T R\,ds + \int_\tau^T \sigma\,dw(s)$$

$$= X(\tau) + R(T-\tau) + \sigma \underbrace{\left(w(T) - w(\tau)\right)}_{w(T-\tau)}$$

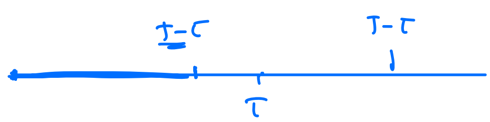
Remark: $w(T) - w(\tau) \sim \mathcal{N}(0, T-\tau)$

$w(T-\tau) \sim \mathcal{N}(0, T-\tau)$

$$X(T) = x(t) + R(T-\tau) + \sigma\,w(T-\tau) \qquad F(T, x) = x$$

$$E\left[\, x(\tau) + R(T-\tau) + \sigma\,w(T-\tau) \mid x(\tau) = x \,\right]$$

$$= x + R(T-t) + \sigma E\left[\, w(T-\tau) \mid x(\tau) = x \,\right]$$



Compute the solution of such problem.

6. Solve the following boundary value problem:

$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2}(t,x) = 0, \quad t \in [0,T], x > 0$$
$$F(T,x) = (\ln(x))^2$$

$\mu = 0$ $\quad \frac{1}{2}\sigma^2$

"auxiliary" process:

$$dx(\tau) = 0\,dt + \sigma x(\tau)\,dw(\tau)$$

$$\Rightarrow x(T) = x(t)\, e^{-\frac{1}{2}\sigma^2(T-\tau) + \sigma(w(T) - w(\tau))}$$

$$F(t,x) = E\left[\left(\ln x(\tau) - \frac{1}{2}\sigma^2(T-t) + \sigma(w(T) - w(\tau))\right)^2 \Big| x(\tau) = x\right]$$

$$= E\left[\left(\underbrace{\ln x - \frac{1}{2}\sigma^2(T-\tau)}_{a} + \sigma(w(T)-w(\tau)\right)^2 \Big| x(\tau) = x\right]$$

$$= E\left[a^2 + \sigma^2(w(T)-w(t))^2 + 2a(w(T)-w(\tau)) \Big| x(\tau) = x\right]$$

$$= \left(\ln x - \frac{1}{2}\sigma^2(T-\tau)\right)^2 + \sigma^2 \underbrace{E[(w(T)-w(\tau))^2]}_{\operatorname{var}(w(T-\tau))} +$$

$$+ 2a\, \cancel{E[w(T)-w(\tau)]}$$

$$= \left(\ln x - \frac{1}{2}\sigma^2(T-t)\right)^2 + \sigma^2(T-\tau)$$

Note:

$$dx(\tau) = \mu x(\tau)\,d\tau + \sigma x(\tau)\,dw(\tau)$$

$$dx(s) = \mu\,ds + \sigma\,dw(s)$$

$$\int_\tau^T dx(s) = \int_\tau^T \mu\,ds + \int_\tau^t \sigma\,dw(s)$$

$$x(T) - x(\tau) = \mu(T-t) + \sigma(w(T) - w(\tau))$$

if $\mu = 0$ and $\sigma = 1$: $\quad dx(\tau) = dw(\tau)$

b) $\frac{\partial u}{\partial t} + \alpha(\theta - x)\frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = 0$, with $u(x,T) = F(x)$

$\sigma = 1$

Solution: $x(T) = \theta + (x(\tau) - \theta)\, e^{-\alpha(T-\tau)} + \int_{\tau}^{T} e^{-\alpha(T-s)}\, dw(s)$

$\left[dx(\tau) = \alpha(\theta - x(\tau))\, d\tau + dw(\tau) \right]$

5. Consider the following boundary value problem:

$$\frac{\partial F}{\partial t} \overset{\mu}{\boxed{-\frac{1}{1-t}}}\frac{\partial F}{\partial x} + \overset{\frac{1}{2}\sigma^2}{\boxed{\frac{1}{2}}}\frac{\partial^2 F}{\partial x^2} = 0$$
$$F(T,x) = x^2$$

[$dX(\tau) = -\frac{1}{1-\tau}\, d\tau + dW(\tau)$

solution: $X(\tau) = -\int_0^\tau \frac{1}{1-s}\, ds + W(\tau)$

$= \ln(1-\tau) + W(\tau)$

$dX(\tau) = -\frac{1}{1-\tau} X(\tau)\, d\tau + dW(\tau)$

solution: $X(\tau) = (1-\tau)\int_0^\tau \frac{1}{1-u}\, dW(u)$

$\mu = -\frac{1}{1-t}$ $\qquad \sigma = 1$

$$dX(s) = -\frac{1}{1-s}\, ds + dW(s)$$

$$\int_t^T dX(s) = -\int_t^T \frac{1}{1-s}\, ds + \int_t^T dW(s)$$

$$X(T) - X(t) = +\int_t^T (\ln(1-s))'\, ds + \int_t^T W(s)$$

$$\Rightarrow \quad X(T) = X(t) + \ln(1-T) - \ln(1-t) + W(T) - W(t)$$

$$F(t,x) = E\left[X(t) + \ln\left(\frac{1-T}{1-t}\right) + W(T) - W(t)) \,\middle|\, X(t) = x \right]$$
$$= x + \ln\left(\frac{1-T}{1-t}\right) + E[W(T)\cancel{-W(t)}]$$

Another exercise:

$$\frac{\partial F}{\partial t} - \frac{1}{1-t}\, x \frac{\partial F}{\partial x} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} = 0$$

$$dX(\tau) = -\frac{1}{1-\tau} X(\tau)\, d\tau + dW(\tau)$$

Solution: $X(\tau) = (1-\tau)\int_0^\tau \frac{1}{1-u}\, dW(u)$

$$X(T) = (1-T)\int_0^T \frac{1}{1-u}\, dW(u)$$

$$= (1-T)\,\frac{(1-\tau)}{1-\tau}\left[\int_0^{\tau}\frac{1}{1-u}\,dw(u) + \int_{\tau}^{1}\frac{1}{1-u}\,dw(u)\right]$$

$$= \frac{1-T}{1-\tau}\Big[\underbrace{(1-\tau)\int_0^{\tau}\frac{1}{1-u}\,dw(u)}_{X(\tau)} + \int_{\tau}^{t}\frac{1}{1-u}\,dw(u)\Big]$$

$$\Rightarrow \quad X(t) = \frac{1-T}{1-\tau}X(\tau) + \frac{1-T}{1-\tau}\int_{\tau}^{T}\frac{1}{1-u}\,dw(u)$$

$$F(T, x) = x$$

$$F(t, x) = E\left[\frac{1-T}{1-\tau}X(\tau) + \frac{1-T}{1-\tau}\int_{\tau}^{t}\frac{1}{1-u}\,dw(u)\,\Big|\,X(\tau)=x\right]$$

$$= \frac{1-T}{1-\tau}x + \frac{1-T}{1-\tau}E\Big[\underbrace{\cancel{\int_{\tau}^{t}\frac{1}{1-u}\,dw(u)}}_{\text{"}0}\Big]$$

1. Consider the usual Black-Scholes model but suppose that we introduce a new contract, whose payoff at maturity date T is equal to $S^2(T)$. Derive that the arbitrage-free pricing at time t.
   Note: $\mathbb{E}[e^{tX}] = e^{t\mu+\frac{1}{2}\sigma^2 t}, \quad X \sim \mathcal{N}(\mu, \sigma^2)$.

$$\Pi(0; S(0)) = e^{-RT}E^{Q}[S^2(T)\,|\,S(0)]$$

$$S(T) = S(0)\,e^{(R-\frac{1}{2}\sigma^2)T + \sigma W(T)}$$

$$S^2(T) = S^2(0)\,e^{2(R-\frac{1}{2}\sigma^2)T + 2\sigma W(T)}$$

$$\Pi(0; S(0)) = e^{-RT}S^2(0)\,e^{2(R-\frac{1}{2}\sigma^2)T}\,E\big[e^{\underbrace{2\sigma}_{t}\underbrace{W(T)}_{X}}\big]$$

$$W(T) \sim \mathcal{N}(0, T)$$

$$E[e^{tX}] = e^{t\mu + \frac{1}{2}\sigma^2 t} \qquad X \sim \mathcal{N}(\mu, \sigma^2)$$

$$= e^{(R-\sigma^2)T}\,S^2(0)\,e^{2\sigma\times 0 + \frac{1}{2}(2\sigma)T}$$

$$= S^2(0)\,e^{(R-\sigma^2+\sigma)T}$$

12. Consider the Black-Scholes model, with the usual assumptions. Derive the price at time $t < T$ years (where T is the maturity) of a call option with contract function:

$$g(x) = \begin{cases} \cancel{0}\ a & x \le K \\ \cancel{1}\ b & x > K \end{cases}$$

where $K > 0$. Assume a continuously compounded interest rate equal to r per year. In particular, provide the price of this option at time 0 when $K = S(0)$.

$$\Pi(0; S(0)=x) = e^{-RT} E^{Q}\left[g(S(T)) \mid S(0)=x \right]$$

Initially: $S(T) = S(0)\, e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W(T)}$

Under Q : $S(T) = S(0)\, e^{(R - \frac{1}{2}\sigma^2)T + \sigma W(T)}$

$$\Pi(0; S(0)=x) = e^{-RT}\left[a\, P(S(T) \le K \mid S(0)=x) + b\, P(S(T) > K \mid S(0)=x) \right] = \otimes$$

$$\left(E[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x)\,dx = \int_{-\infty}^{K} a f_X(x)\,dx + \int_{K}^{+\infty} b f_X(x)\,dx = a\,P(X \le K) + b\,P(X > K) \right)$$

$$P(S(T) \le K \mid S(0)=x) = P\left(x\, e^{(R-\frac{1}{2}\sigma^2)T + \sigma \underline{W(T)}} \le K \right)$$

$$= P\left(e^{(R-\frac{1}{2}\sigma^2)T + \sigma W(T)} \le \frac{K}{x} \right)$$

$$= P\left((r - \tfrac{1}{2}\sigma^2)T + \sigma W(T) \le \ln \frac{K}{x} \right)$$

$$= P\left(W(T) \le \frac{\ln \frac{K}{x} - (r - \frac{1}{2}\sigma^2)T}{\sigma} \right) =$$

$W(T) \sim N(0,T)$

$\frac{W(T)}{\sqrt{T}} \sim N(0,1)$

$$= \Phi\left(\frac{\ln \frac{K}{x} - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right)$$

$$= F_{N(0,T)}\left(\frac{\ln \frac{K}{x} - (r - \frac{1}{2}\sigma^2)T}{\sigma} \right)$$

Thus

$$\Pi(0;\ S(0)=x) = e^{-rT}\Bigg[a\,\Phi\left(\frac{\ln\frac{K}{x} - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) +$$

$$+\, b\,\Bigg(1 - \Phi\left(\frac{\ln\frac{K}{x} - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

<u>Note</u> :

$$P(X \le x) = F_X(x) = P(X<x) \quad \text{(continuous r.v)}$$

$$P(X \ge x) = 1 - F_X(x) = P(X>x)$$

$$\Big)$$

11. Consider the following derivative X: the buyer of X obtains, at the maturity T, the value $\ln(S(T))$, where S is the stock price process. Derive its arbitrage free price and comment the result, notably in view of the possible values of the payoff of this product.